Superconducting Topological Fluids in Josephson Junction Arrays

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We argue that frustrated Josephson junction arrays may support a topologically ordered superconducting ground state, characterized by a non-trivial ground state degeneracy on the torus. This superconducting quantum fluid provides an explicit example of a system in which superconductivity arises from a topological mechanism rather than from the usual Landau-Ginzburg mechanism.

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It is by now widely believed that quantum phase transitions describe changes in the entanglement pattern of the complex-valued ground state wave function and that the universality classes of these quantum ground states define the corresponding quantum orders [1]. When there is a gap in the spectrum the quantum ordered ground state is called topologically ordered [2]; its hallmark is a ground state degeneracy depending only on the topology of the underlying space.

The best known example of topological order is given by Laughlin's quantum incompressible fluids [3] describing the ground states responsible for the quantum Hall effect [4]. Soon after Laughlin's discovery, it was conjectured that an analogous mechanism could enable superconductivity [5] in high- temperature or granular superconductors, although the original anyon superconductivity mechanism has now been ruled out experimentally, due to the observed parity and time reversal invariance of the ground states of the relevant physical systems.

Josephson junction arrays (JJA) have been regarded by several authors [6], [7], [8], [9] as controllable devices, which may exhibit topological order. Planar arrays display a characteristic insulator-superconductor quantum phase transition at $T = 0$ [10]. In [8] we pointed out that two-dimensional JJAs may be mapped onto an Abelian gauge theory with Chern-Simons term and evidenced how the topological gauge theory, which naturally allows for the appearance of topological order, together with duality may be useful to describe the phase diagram of these devices.

The gauge theory formulation of JJAs [8] clearly evidences that the superconducting ground state is a P- and T- invariant generalization of Laughlin's incompressible quantum fluid. The simplest example of a topological fluid [11] is a ground-state described by a low-energy effective action given solely by the topological Chern-Simons term [12] $S = k/4\pi \int d^3x \ A_\mu \epsilon^{\mu\nu\alpha} \partial_\nu A_\alpha$ for a compact $U(1)$ gauge field A_μ whose dual field strength $F^\mu = \epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha$ yields the conserved matter current. In this case the degeneracy of the ground state on a manifold of genus g will be k^g (or $(k_1 k_2)^g$ if $k = k_1/k_2$ is a rational number): for planar unfrustrated JJAs one finds that the topological fluid is described by two $k = 1$ Chern-Simons gauge fields of opposite chirality and, thus, there is no degeneracy of the ground state [8], [13].

In this letter we argue that frustrated JJAs may support a topologically ordered ground state described by a pertinent superconducting quantum fluid, thus providing an interesting and explicit example of a system in which superconductivity arises from the topological mechanism proposed in [13] rather than from the usual Landau-Ginzburg mechanism. In presence of n_q offset charge quanta per site and n_ϕ external magnetic flux quanta per plaquette in specific ratios, Josephson junction arrays might support incompressible quantum fluid [14] phases corresponding to purely two-dimensional quantum Hall phases for either charges [15] or vortices [16]. In this paper we shall show that, if quantum Hall phases for charges or vortices are realized, then JJA naturally support a topologically ordered ground state and a phase in which they behave as a topological superconductor [13]; there is, in fact, a renormalization of the Chern-Simons coefficient yielding a non-trivial ground state degeneracy on the torus (and in general on manifolds with non-trivial topology).

We shall consider JJAs fabricated on a square planar lattice of spacing $l = 1$ made of superconducting islands with nearest neighbours Josephson couplings of strength E_J [10]. Each island has a capacitance C_0 to the ground; moreover, there are nearest neighbours capacitances C. To implement a torus topology we impose doubly periodic conditions at the boundary of the square lattice.

In [8] we have shown that the zero-temperature par-

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tition function of JJAs may be written in terms of two effective gauge fields A_μ (vector) and B_μ (pseudovector). In the low energy limit the partition function is

$$
\begin{aligned}
Z &= \sum_{\substack{\{Q_0\}\\ \{M_0\}}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \, \exp(-S) \ , \\
S &= \int dt \sum_{\mathbf{x}} -i \frac{1}{2\pi} \, A_\mu K_{\mu\nu} B_\nu + i A_0 Q_0 + i B_0 M_0 \quad (1)
\end{aligned}
$$

This form of the partition function holds true also with toroidal boundary conditions. The gauge fields embody the original degrees of freedom through their dual field strengths $q_\mu \propto K_{\mu\nu}B_\nu$ and $\phi_\mu \propto K_{\mu\nu}A_\nu$ representing, respectively, the conserved charge (vector) current and the conserved vortex (axial) current. $K_{\mu\nu}$ is the lattice Chern-Simons term [17], defined by $K_{00} = 0$, $K_{0i} = -\epsilon_{ij} d_j$, $K_{i0} = S_i \epsilon_{ij} d_j$ and $K_{ij} = -S_i \epsilon_{ij} \partial_0$, in terms of forward (backward) shift and difference operators S_i ($\hat{S}_i$) and d_i ($\hat{d}_i$). Its conjugate $\hat{K}_{\mu\nu}$ is defined by $\hat{K}_{00} = 0$, $\hat{K}_{0i} = -\hat{S}_i \epsilon_{ij} \hat{d}_j$, $\hat{K}_{i0} = \epsilon_{ij} \hat{d}_j$ and $\hat{K}_{ij} = -\hat{S}_j \epsilon_{ij} \partial_0$. The two Chern-Simons kernels $K_{\mu\nu}$ and $\hat{K}_{\mu\nu}$ are interchanged upon integration (summation) by parts on the lattice.

The topological excitations are described by the integer-valued fields Q_0 and M_0 and represent unit charges and vortices rendering the gauge field components A_0 and B_0 integers via the Poisson summation formula; their fluctuations determine the phase diagram [8]. In the classical limit the magnetic excitations are dilute and the charge excitations condense rendering the system a superconductor: vortex confinement amounts here to the Meissner effect. In the quantum limit, the magnetic excitations condense while the charged ones become dilute: the system exhibits insulating behavior due to vortex superconductivity accompanied by a charge Meissner effect.

By rewriting the topological excitations as the curl of an integer-valued field

$$
\begin{aligned}
Q_0 &\equiv K_{0i} Y_i \ , & Y_i \in Z \ , \\
M_0 &\equiv \hat{K}_{0i} X_i \ , & X_i \in Z \ , \qquad (2)
\end{aligned}
$$

we get the mixed Chern-Simons term as follows:

$$
\begin{aligned}
Z &= \textstyle\sum_{\{X_i, Y_i\}} \int \mathcal{D}A_\mu \int \mathcal{D}B_\mu \, \exp\left(-S\right) \ , \\
S &= -\tfrac{1}{2\pi} i \int dt \textstyle\sum_{\mathbf{x}} A_0 K_{0i} \left(B_i - 2\pi Y_i\right) \\
&\quad + B_0 \hat{K}_{0i} \left(A_i - 2\pi X_i\right) + A_i K_{ij} B_j \ . \qquad (3)
\end{aligned}
$$

From (3) one sees that the gauge field components A_i and B_i are angular variables due to their invariance under time-independent integer shifts. Such shifts do not affect the last term in the action, which contains a time derivative, and may be reabsorbed in the topological excitations X_i and Y_i, leaving also the first term of the action invariant. The low energy theory is thus compact.

In analogy with the conventional quantum Hall setting one should expect the charge and vortex transport properties to depend on the ratios of the offset charges (i.e. the filling fractions) (n_q/n_ϕ) and (n_ϕ/n_q), respectively. Due to the periodicity of the charge-vortex coupling, however, n_ϕ (n_q) is defined only modulo an integer as far as charge (vortex) transport properties are concerned. Using this freedom one may define effective filling fractions (we shall assume $n_q \geq 0$, $n_\phi \geq 0$ for simplicity) as

$$
\begin{aligned}
\nu_q &\equiv \tfrac{n_q}{n_\phi - [n_\phi]^- + [n_q]^+} \ , & 0 \leq \nu_q \leq 1 \ , \\
\nu_\phi &\equiv \tfrac{n_\phi}{n_q - [n_q]^- + [n_\phi]^+} \ , & 0 \leq \nu_\phi \leq 1 \ , \qquad (4)
\end{aligned}
$$

where $[n_q]^\pm$ indicate the smallest (greatest) integer greater (smaller) than n_q. Of course, these effective filling fractions are always smaller than 1.

In [8] we assumed the existence of these quantum Hall phases and discussed them in the framework of the gauge theory representation of Josephson junction arrays, showing that, depending on certain parameters of the array there are both a charge quantum Hall phase and a vortex quantum Hall phase. Here we will concentrate on the low energy limit of the charge quantum Hall phase and we will show that the system has topological order and behaves as a superconductor when charge condenses.

The pertinent low energy theory is now given by:

$$
S = \int dt \sum_{\mathbf{x}} -\frac{i}{\pi} A_\mu K_{\mu\nu} B_\nu - \frac{i\nu_q}{\pi} A_\mu K_{\mu\nu} A_\nu \ , \qquad (5)
$$

with $\nu_q = p/n$. The main difference with (1) is the addition of a pure Chern-Simons term for the A_μ gauge field . We have also rescaled the coefficient of the mixed Chern-Simons coupling by a factor of 2 (compare with (1)). This factor of 2 is a well-known aspect of Chern-Simons gauge theories [18]. Moreover, since in JJAs the charge degrees of freedom are bosons, the allowed [8] filling fractions are given by $\nu_q = \frac{p}{n}$, with $pn =$ even integer in accordance with [19]. As a result, the action (5) may now be written in terms of two independent gauge fields A_μ and $B^q_\mu = B_\mu + \nu_q A_\mu$ yielding:

$$
S = \int dt \sum_{\mathbf{x}} -\frac{i}{\pi} A_\mu K_{\mu\nu} B^q_\nu \ . \qquad (6)
$$

In describing JJAs one has to require the periodicity of charge-vortex couplings; the coupling of the topological excitations enforcing the periodicity of the mixed Chern-Simons term $A_\mu K_{\mu\nu} B^q_\nu$ is then:

$$
S = \int dt \sum_{x} \cdots + i p A_0 Q_0 + i n B_0 M_0 \ , \qquad (7)
$$

that can be rewritten as:

$$
S = \sum_{x} \cdots + i l p A_0 \left(Q_0 + M_0\right) + i l n B^q_0 M_0 \ . \qquad (8)
$$

Due to the replacement $B_\mu \to B^q_\mu$, the periodicities of the two original gauge fields are

$$\begin{aligned} A_i &\to A_i + \pi n\ a_i\ , \qquad & a_i \in Z\ , \\ B_i &\to B_i + \pi p\ b_i\ , \qquad & b_i \in Z\ , \end{aligned} \tag{9}$$

and

$$B^q_i \to B^q_i + \pi p\ b_i\ , \qquad b_i \in Z\ . \tag{10}$$

The resulting low energy theory is thus, again, compact.

Using the representation (2), one may rewrite the mixed Chern-Simons term as

$$S = \int dt \sum_x \ldots \ -i\tfrac{(pq/2)}{2\pi}\left(\tfrac{2A_0}{n}\right) K_{0i}\left(\tfrac{2B^q_i}{p} - 2\pi Y_i\right) \\ -i\tfrac{(pq/2)}{2\pi}\tfrac{2B^q_0}{n} K_{0i}\left(\tfrac{2A_i}{n} - 2\pi X_i\right)\ . \tag{11}$$

In this representation it is clear that the topological excitations render the charge-vortex coupling periodic under the shifts

$$\begin{aligned} A^{'}_i &= \frac{2A_i}{n} \ \to A^{'}_i + 2\pi\ a_i\ , \qquad & a_i \in Z\ , \\ B^{'}_i &= \frac{2B^q_i}{p} \ \to B^{'}_i + 2\pi\ b_i\ , \qquad & b_i \in Z\ . \end{aligned} \tag{12}$$

This model corresponds to two Chern-Simons terms with coefficients $\pm k/4\pi$ with $k = np/2$ an integer. It is worth to point out that, since B^q_ν does not have a definite parity (is a liner combination of a vector and a pseudovector) the model is not P- and T-invariant, as it must be due to the presence of the Chern-Simons term for the field A_μ.

The hallmark of topological order is the degeneracy of the ground state on manifolds with non-trivial topology as shown by Wen [2]. The torus degeneracy on the lattice of the Chern-Simons model was computed in [20]. For a single Chern-Simons term this degeneracy is $(k)^g$ where k is the integer coefficient of the Chern-Simons term, and g the genus of the surface. In our case this degeneracy is $2 \times (k)^g = 2 \times \frac{np}{2}$, since we have two Chern-Simons terms. This degeneracy is exactly what is expected for a doubled Chern-Simons model [21], for which the physical Hilbert space is the direct product of the two Hilbert spaces of the component models.

We will now demonstrate that the phase where topological excitations Q_0 condense while M_0 are dilute describes an effective gauge theory of a superconducting state. The partition function is:

$$\begin{aligned} Z_{LE} &= \sum_{\{Q_0\}} \int \mathcal{D}A_\mu \int \mathcal{D}B^q_\mu \ \exp\left(-S\right)\ , \\ S &= \int dt \sum_{\mathbf{x}} -\frac{ik}{2\pi}\ A_\mu K_{\mu\nu} B^q_\nu + \frac{ik}{2\pi} A_0 (2\pi Q_0) \end{aligned} \tag{13}$$

To this end note first that a unit external charge, represented by an additional term $i2\pi a_0(t,\mathbf{x})\delta_{\mathbf{x}\mathbf{x_0}}$ is completely screened by the charge condensate, since it can be absorbed into a redefinition of the topological excitations Q_0. In order to characterize the superconducting phase we introduce the typical order parameter namely the 't Hooft loop of length T in the time direction:

$$L_H \equiv \exp\left(i\phi\frac{\kappa}{2\pi}\int dt \sum_x \phi_\mu B_\mu\right)\ , \tag{14}$$

where $\phi_0(t,\mathbf{x}) = \left(\theta\left(t+T/2\right) - \theta\left(t-T/2\right)\right)\delta_{\mathbf{x}\mathbf{x_1}}$ - $\left(\theta\left(t+T/2\right) - \theta\left(t-T/2\right)\right)\delta_{\mathbf{x}\mathbf{x_2}}$ and $\phi_i(-T/2,\mathbf{x})$, $\phi_i(T/2,\mathbf{x})$ are unit links joining $\mathbf{x_1}$ to $\mathbf{x_2}$ and $\mathbf{x_2}$ to $\mathbf{x_1}$ at fixed time and vanishing everywhere else. Its vacuum expectation value $\langle L_H \rangle$ yields the amplitude for creating a separated vortex-antivortex pair of flux ϕ, which propagates for a time T and is then annihilated in the vacuum.

Since we replaced $B_\mu \to B^q_\mu$, we may rewrite the 't Hooft loop as:

$$L_H \equiv \exp\left(i\phi\frac{\kappa}{2\pi}\int dt \sum_x \left(\phi_\mu B^q_\mu - \frac{p}{n}A_\mu\phi_\mu\right)\right)\ . \tag{15}$$

To compute $\langle L_H \rangle$ one should integrate first over the gauge field B^q_μ to get

$$\begin{aligned} \langle L_H \rangle \quad &\propto \sum_{\{Q_0\}} \int \mathcal{D}A_\mu \ \delta\left(\hat{K}_{\mu\nu}A_\nu - \phi\phi_\mu\right) \times \\ & \exp i\phi\frac{\kappa}{2\pi}\left(\int dt \sum_{\mathbf{x}} A_0(2\pi Q_0) - \frac{p}{n}A_\mu\phi_\mu\right) \end{aligned} \tag{16}$$

The sum over Q_0 enforces the condition that $\frac{k}{2\pi}A_0$ be an integer. As a consequence, $\hat{K}_{i0}A_0 = \frac{2\pi}{k}n_i = \phi\phi_i$ with n_i an integer. We thus have:

$$\phi = \frac{2\pi}{k}q, q \in Z\ ; \tag{17}$$

thus, $\langle L_H \rangle$ vanishes for all fluxes different from an integer multiple of the fundamental fluxon, which is just the Meissner effect. In the low-energy effective gauge theory vortex-antivortex pairs are confined by an infinite force which becomes logarithmic upon including also higher-order Maxwell terms.

From (11) one has (depending if p is an even integer or n is an even integer) either that n is the charge unit with $p/2$ units of charge, or, viceversa, that p is the charge unit with $n/2$ units of charge. By rewriting (17) as $\phi = \frac{2}{pn}2\pi n$ one finds the standard flux quantization $\phi = \frac{2\pi}{Ne}n$ where e is the charge unit and N is the number of units of charge.

Summarizing, frustrated planar JJAs in the quantum Hall phases provide an explicit example of both topological order and of a new superconducting behavior [13] analogous to Laughlin's quantum Hall fluids.

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